

AD-A117 774 FLORIDA UNIV GAINESVILLE CENTER FOR MATHEMATICAL SYS--ETC F/G 12/1
FURTHER RESULTS ON POLYNOMIAL CHARACTERIZATIONS OF (F,G)-INVARI--ETC(U
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 15077.15-MA	2. GOVT ACCESSION NO. AD-A117774	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) Further Results on Polynomial Characterizations of (F,G)-Invariant and Reachability Subspaces		5. TYPE OF REPORT & PERIOD COVERED Reprint
7. AUTHOR(s) Pramod P. Khargonekar Erol Emre		6. PERFORMING ORG. REPORT NUMBER N/A
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Florida Gainesville, FL 32611		8. CONTRACT OR GRANT NUMBER(s) DAAG29 80 C 0050
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12011 Research Triangle Park, NC 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE Apr 82
		13. NUMBER OF PAGES 15
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Submitted for announcement only.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) B		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		

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FURTHER RESULTS ON POLYNOMIAL CHARACTERIZATIONS
OF (F, G) -INVARIANT AND REACHABILITY SUBSPACES

Pramod P. Khargonekar and Erol Emre

Reprinted from IEEE Transactions on Automatic Control, Vol. AC-27, No. 2, April 1982
0018-9286/82/0400-0352 \$00.75 © 1982 IEEE

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Further Results on Polynomial Characterizations of (F, G) -Invariant and Reachability Subspaces

PRAMOD P. KHARGONEKAR AND EROL EMRE, MEMBER, IEEE

Abstract—This paper is concerned with further development of the unification between polynomial matrix and geometric theories of linear systems following the work of Emre and Hautus. Equivalence between different polynomial characterizations of (F, G) -invariant and reachability subspaces is shown explicitly. Several new results are given which clarify the relations between the polynomial system matrix, invariant subspaces, and system zeros. Finally, a polynomial characterization of and a constructive procedure to obtain the largest stabilizability subspace in $\ker H$ are given.

I. INTRODUCTION

IN recent years two of the main approaches to algebraic linear control theory for finite dimensional linear systems defined over fields are the "polynomial matrix approach" (see Rosenbrock [21], Wolovich [24], and the references therein) and the "geometric approach" (see Wonham [25] and the references given there). Solutions which are apparently unrelated have been given to several

control problems with both approaches. The main fundamental concepts for the "geometric" theory are (F, G) -invariant and reachability subspaces (see Basile and Marro [2] and Wonham [25]), whereas for the "polynomial matrix approach" the main fundamental concepts are the matrix fraction representations and the associated system matrices (see Rosenbrock [21] and Wolovich [24]).

A systematic unification of these apparently disjointed approaches is necessary for a unified study and a better understanding of these problems as well as for possible extensions of these results to more general classes of systems. Although previously several authors have established some equivalences between these two approaches for several purposes (see Emre [3], [5], Fuhrmann [7], [8], Moore and Silverman [19], Morse [20], and the references therein), the first systematic approach towards a unification has started with Emre and Hautus [6], where polynomial characterizations of (F, G) -invariant and reachability subspaces have been given in terms of matrix fraction descriptions and system matrices. This approach has been based on a natural realization of matrix fraction descriptions introduced by Fuhrmann [7], [8], which follows the module theoretic realization theory of Kalman [14] (see Kalman, Falb, and Arbib [15, ch. 10] for a detailed exposition of this approach and Emre [4] for a simple derivation of the realization of Fuhrmann [7], [8] based on the input output map introduced by Kalman [14]). Later, this line of research has been continued by Khargonekar and Emre [16], Fuhrmann

Manuscript received June 27, 1980; revised April 22, 1981 and July 13, 1981. Paper recommended by B. Francis, Past Chairman of the Linear Systems Committee. This work was supported in part by the U.S. Army under Research Grant DAAG29-80-C-0050 and the U.S. Air Force under Grant AFOSR 76-3034 Mod. D, through the Center for Mathematical System Theory, University of Florida, Gainesville, FL. This paper is an extended version of the paper "A Structure Theorem for Polynomial Matrices and (F, G) -Invariant Subspaces," by P. P. Khargonekar and E. Emre, June 1979.

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and Willems [10], and Fuhrmann [9], considering polynomial representations and by Hautus [11], [12], considering rational matrix representations.

The purpose of this paper is to present further results on the polynomial characterizations of (F, G) -invariant and reachability subspaces mainly following Emre and Hautus [6], and also establish an explicit connection to Fuhrmann and Willems [10].

In Section II, we introduce some notation and preliminary results that we will need in the later sections.

In Section III, we first establish explicitly the equivalence of the polynomial characterizations of (F, G) -invariant subspaces given in Emre and Hautus [6] and Fuhrmann and Willems [10] for the case treated in the latter paper. In the latter paper a polynomial characterization is given mainly for the case where the transfer matrix is described as PQ^{-1} where P and Q are polynomial matrices. Using the results given in [6] for the general case where the transfer matrix is allowed to be in the form $PQ^{-1}R$, we establish that the characterizations given in [10] are essentially the same as those of Emre and Hautus [6] specialized to the PQ^{-1} case. Then again for the PQ^{-1} case we establish a correspondence between (F, G) -invariant subspaces and the pairs of matrices (F_1, H_1) . This then is used to obtain a parametrization of (F, G) -invariant subspaces for the PQ^{-1} case. Then, for a given (F, G) -invariant subspace Ψ , we present a polynomial characterization of all feedback matrices L such that Ψ is an $(F + GL)$ -invariant subspace. Further, also using the relation between (F, G) -invariant subspaces and nonsingular factors of polynomial matrices established in Emre [4], we show explicitly for the first time that introducing common factors in the matrix fraction descriptions of the form PQ^{-1} (thus cancelling some system zeros) is the same as making an (F, G) -invariant subspace in $\ker H$ unobservable by state feedback. This extends and clarifies the research started in Wolovich [23] where it is shown that common factors can be introduced in matrix fraction descriptions of the form PQ^{-1} by state feedback.

In Section IV, we consider a matrix fraction description $PQ^{-1}R$ and the associated system matrix T . Then we establish an explicit module isomorphism between the largest (F, G) -invariant and reachability subspaces in $\ker H$, where (F, G, H) is the natural realization associated with T (see Fuhrmann [7], [8], Emre [4], and Section II), and those of a simpler observable system in terms of the system matrix using the results of Emre and Hautus [6]. This also provides a constructive procedure to obtain the largest reachability subspace in $\ker H$ for the $PQ^{-1}R$ case, for the first time.

In Section V, based on the results of Section IV and those of Emre and Hautus [6], and introducing a module structure on the largest (F, G) -invariant and reachability subspaces in $\ker H$, we establish a theorem on the invariant factors of a general polynomial matrix which generalizes and unifies both a theorem of Fuhrmann [7] on the invariant factors of a nonsingular polynomial matrix and a theorem of Moore and Silverman [19] on transmission polynomials which was later reproved by Anderson [1] and generalized by Molinari [18].

Finally, in Section VI we present a polynomial characterization of stabilizability subspaces which play an important role in a number of control problems (see Wonham [25] and Hautus [11], [12]). Our results on polynomial characterization of stabilizability subspaces also lead to a new constructive procedure to obtain the largest stabilizability subspace contained in $\ker H$ in terms of the system matrix.

II. NOTATION AND PRELIMINARY RESULTS

In this section we introduce some notation, some preliminary definitions, and results that we will need in the sequel.

Let K be a field. Let $K[z]$ denote the ring of polynomials in z with coefficients in K and let $K(z)$ denote the field of formal Laurent series in the indeterminate z^{-1} with coefficients in K . If S is any given set and p, m are positive integers, then S^p denotes the set of p -vectors and $S^{p \times m}$ denotes the set of $p \times m$ matrices with entries in S . If x is an element of $K^p(z)$ then $(x)_i$ denotes the polynomial part of x , $(x)_{-1}$ denotes the coefficient of z^{-1} in the formal Laurent series expansion of x in z^{-1} , and $(x)^+$ denotes the strictly proper part of x , i.e.,

$$(x)^+ = x - (x)_0.$$

For a $p \times m$ matrix A whose columns belong to a K -linear space V , $Sp_K A$ denotes K -linear space spanned by the columns of V . If V is also a $K[z]$ -module then $\langle A \rangle$ denotes the $K[z]$ -submodule generated by the columns of A . Furthermore, if f is a function with V as its domain and if a_i denotes the i th column of A , then $f(A)$ denotes the matrix whose i th column is $f(a_i)$. Finally, if f is a function then $\text{im } f$ denotes the image of f , and if f is a linear mapping then $\ker f$ denotes the kernel of f .

Let T be a $p \times m$ polynomial matrix. Then K_T is defined as

$$K_T = \{x: x \text{ belongs to } K^p[z] \text{ and there exists a strictly proper vector } q \text{ such that } x = Tq\}.$$

In particular, if T is a square and nonsingular $p \times p$ polynomial matrix, then we have

$$K_T = \{x: x \text{ belongs to } K^p[z] \text{ and } T^{-1}x \text{ is a strictly proper vector}\}$$

and the map π_T is defined as

$$\pi_T: K^p[z] \rightarrow K_T: x \mapsto T(T^{-1}x)^+.$$

Let Z be a $p \times m$ strictly proper transfer matrix with matrix fraction description

$$Z = PQ^{-1}R_1$$

where Q, P , and R_1 are $r \times r, p \times r$, and $x \times m$ polynomial matrices, respectively. Let T be the corresponding system matrix (see Rosenbrock [21])

$$T = \begin{bmatrix} Q & R_1 \\ -P & 0 \end{bmatrix}.$$

Throughout the paper our results will be in terms of the system matrix T . For convenience we will assume that $Q^{-1}R_1$ is strictly proper without loss of generality, because if $Q^{-1}R_1$ is not strictly proper then we will define

$$R := \pi_Q(R_1)$$

and we will then use the associated system matrix

$$T = \begin{bmatrix} Q & R \\ -P & U \end{bmatrix}$$

where U is the unique $p \times m$ polynomial matrix such that

$$Z = PQ^{-1}R + U.$$

Define the K -linear maps

$$\begin{aligned} F_Q: K_Q &\rightarrow K_Q: x \mapsto \pi_Q(zx) \\ G_Q: K^m &\rightarrow K_Q: u \mapsto Ru \end{aligned}$$

and

$$H_Q: K_Q \rightarrow K^p: x \mapsto (PQ^{-1}x)_1.$$

The following lemma associates a natural realization of Z with the system matrix T .

Lemma 2.1 [7]: Let Z be a $p \times m$ strictly proper transfer matrix. Let P, Q, R , and U be polynomial matrices such that

$$Z = PQ^{-1}R + U$$

where $Q^{-1}R$ is strictly proper. Then $\Sigma_Q = (F_Q, G_Q, H_Q)$ is a realization of Z with the state space K_Q . Furthermore, Σ_Q is reachable if and only if Q and R are left coprime and Σ_Q is observable if and only if P and Q are right coprime.

Σ_Q is called the Q -realization of the transfer matrix Z . The following lemma is a slightly different restatement of a result in Emre and Hautus [6].

Lemma 2.2 [6, Theorems 2.5, 2.8]: Let (H, F) be a given observable pair of matrices over the field K . Let Q and S be a pair of left coprime polynomial matrices. Then we have

$$H(zI - F)^{-1} = Q^{-1}S$$

if and only if

- i) the columns of the polynomial matrix S constitute a basis for the K -linear space K_Q and
- ii) the K -linear map

$$\hat{S}: K^n \rightarrow K_Q: x \mapsto Sx$$

provides an isomorphism between the pairs (F, H) and (F_Q, H_Q) (i.e., $F_Q \hat{S} = SF$ and $H_Q \hat{S} = H$).

We will also need the following result in the sequel.

Lemma 2.3 [6, Lemma 3.13]: Let A be a polynomial matrix and let (F, G) be a reachable pair of matrices over K . Then $A(zI - F)^{-1}G$ is a polynomial matrix if and only if $A(zI - F)^{-1}$ is a polynomial matrix.

The following two lemmas give polynomial characterizations of (F_Q, G_Q) -invariant subspaces and reachability subspaces of the Q -realization of Z obtained by Emre and

Hautus [6]. The reader is referred to Wonham [25] for definitions and other important properties of these subspaces. Define the map π as

$$\pi: K^{r \times p}[z] \rightarrow K^r[z]: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto x.$$

Lemma 2.4 [6, sect. 3, 8]: Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization of the $p \times m$ strictly proper transfer matrix

$$Z = PQ^{-1}R + U.$$

Let Ψ be a given $r \times q$ polynomial matrix. Then $Sp_K \Psi$ is an (F_Q, G_Q) -invariant subspace if and only if there exist constant matrices \hat{H}_1, \hat{H}_2 , and F_1 such that

$$Q\hat{H}_1 + R\hat{H}_2 = \Psi(zI - F_1).$$

Furthermore, $Sp_K \Psi$ is contained in $\ker H_Q$ if and only if there exists a $p \times q$ polynomial matrix Φ such that

$$-P\hat{H}_1 + U\hat{H}_2 = \Phi(zI - F_1).$$

If the columns of the polynomial matrix Ψ are K -linearly independent then there exists a linear map L_Q

$$L_Q: K_Q \rightarrow K^m$$

such that

$$(F_Q + G_Q L_Q)\Psi = \Psi F_1$$

(i.e., F_1 is the matrix representation of $(F_Q + G_Q L_Q)$ restricted to $Sp_K \Psi$ with the columns of Ψ taken as a basis for the K -linear space $Sp_K \Psi$.) If T denotes the associated polynomial system matrix of Z then $\pi(K_1)$ is the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$. Furthermore, if $P = I$ then K_R is the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$.

A polynomial characterization of the reachability subspaces is given in the following.

Lemma 2.5 [6, sect. 6]: Let Ψ be a given $r \times q$ polynomial matrix such that the columns of Ψ are K -linearly independent and $Sp_K \Psi$ is an (F_Q, G_Q) -invariant subspace. Let \hat{H}_1, \hat{H}_2 , and F_1 be as in Lemma 2.4. Then

- i) $Sp_K \Psi$ is a reachability subspace if and only if there exists a constant matrix G_1 such that $Sp_K(\Psi G_1) \subset Sp_K R$ and the pair (F_1, G_1) is reachable;
- ii) if G_1 is a constant matrix such that

$$Sp_K \Psi G_1 = Sp_K \Psi \cap Sp_K R$$

then $Sp_K \Psi[G_1, F_1 G_1, \dots, F_1^{q-1} G_1]$ is the largest reachability subspace contained in $Sp_K \Psi$.

We now state the main alternative polynomial characterization of (F_Q, G_Q) -invariant subspaces given by Fuhrmann and Willems [10]. In Section III we will prove the equivalence of the polynomial characterizations of (F_Q, G_Q) -invariant subspaces given by Lemma 2.4 due to Emre and Hautus and the following result due to Fuhrmann and Willems [10].

Lemma 2.6 [10, Lemma 4.2]: Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization of the strictly proper transfer matrix

$$Z := PQ^{-1},$$

Let M be a polynomial matrix such that $Sp_K M$ is a subspace of K_Q . Then $Sp_K M$ is an (F_Q, G_Q) -invariant subspace if and only if there exist polynomial matrices \hat{Q} , Q_1 , and Q_2 such that

$$i) Sp_K M = (Q\hat{Q}^{-1}Q_1K_Q),$$

$$ii) \hat{Q} = Q_1Q_2,$$

iii) $Q\hat{Q}^{-1}$ as well as $\hat{Q}Q^{-1}$ are causal (i.e., $\hat{Q}Q^{-1}$ is bicausal).

The following lemma is a summary of some of the results in Emre [5, Theorems 1, 2].

Lemma 2.7: Let R be a $p \times m$ polynomial matrix. Let Q_1 be a nonsingular $m \times m$ polynomial matrix. Then the following statements are equivalent:

i) Q_1 is a right factor of R .

ii) There exist polynomial matrices V and S_1 (S_1 being left coprime with Q_1) and constant matrices H_1 and F_1 such that

$$Q_1^{-1}S_1 = H_1(zI - F_1)^{-1}$$

and

$$RH_1 = V(zI - F_1).$$

In Section V we will obtain a generalization of a theorem of Moore and Silverman [19]. For the sake of completeness, we now state this theorem.

Theorem 2.8 [19]: Let $\Sigma = (F, G, H)$ be a canonical realization of a strictly proper transfer matrix Z , and let Ψ_M be the largest (F, G) -invariant subspace in $\ker H$ and Ψ_R be the largest reachability subspace in $\ker H$. Then the nonconstant invariant factors of the linear map induced by $(F - GL)$, where L is such that $(F - GL)\Psi_M \subset \Psi_M$, on $\Psi_M - \Psi_R$ are the same as the nonconstant invariant factors of R in a left coprime factorization

$$Z = Q^{-1}R$$

of Z which are the same as the numerator polynomials in the Smith-McMillan form of Z .

III. UNIFICATION OF ALTERNATIVE CHARACTERIZATIONS AND A PARAMETRIZATION OF (F, G) -INVARIANT SUBSPACES

Let F in $K^{n \times n}$ and G in $K^{n \times m}$ be a reachable pair of matrices. Let W and Q be $n \times m$ and $m \times m$ coprime polynomial matrices such that

$$Z := (zI - F)^{-1}G = WQ^{-1}$$

Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization of Z . In this section, first we establish the equivalence of the polynomial characterizations of (F_Q, G_Q) -invariant subspaces obtained by Emre and Hautus [6] and those obtained by Fuhrmann and Willems [10]. These characterizations have been stated in Section II as Lemmas 2.4 and 2.6. The characterization

of (F_Q, G_Q) -invariant subspaces as stated in Lemma 2.6 is the main characterization obtained by Fuhrmann and Willems [10]. They have also considered the case where

$$Z = Q_1^{-1}R_1$$

(with the restriction that Q_1 and R_1 are left coprime). However, the results in this case are obtained using the state-space homomorphism theorem of Fuhrmann [7] relating the Q -realizations corresponding to the coprime factorizations $Q_1^{-1}R_1$ and PQ^{-1} , and using their main result for the case $Z = PQ^{-1}$. Therefore, we restrict our attention to their main characterization given in Lemma 2.6.

Furthermore, for a given (F, G) -invariant subspace Ψ , we give a characterization of all feedback matrices L such that Ψ is $(F - GL)$ -invariant in terms of polynomial matrices. We establish a correspondence between (F_Q, G_Q) -invariant subspaces and pairs of constant matrices (H_1, F_1) , and give a parametrization of these subspaces in terms of the pairs (H_1, F_1) . Finally, for a given strictly proper transfer matrix with the matrix fraction representation PQ^{-1} and the associated Q -realization (F_Q, G_Q, H_Q) , we show for the first time, explicitly, the relation between (F_Q, G_Q) -invariant subspaces in $\ker H_Q$ and the common nonsingular right divisors of P and \hat{Q} where \hat{Q} is feedback equivalent to Q . We give explicit characterization of these factors in terms of (F_Q, G_Q) -invariant subspaces in $\ker H_Q$. This, in particular, shows explicitly that cancelling common factors between P and Q and thus removing some of the system zeros corresponds to making (F_Q, G_Q) -invariant subspaces in $\ker H_Q$ unobservable by state feedback. Previously, partial results on this problem were obtained by Wolovich [23] where it was shown that a nonsingular right divisor of P could be cancelled by state feedback. Here we consider all possible nonsingular right factors of P and show explicitly the relation of (F_Q, G_Q) -invariant subspaces in $\ker H_Q$ to the right nonsingular factors of P , providing a characterization of these factors in terms of these subspaces. These results generalize the research started by Wolovich [23], also making a connection to the geometric theory of linear systems.

It is well known (see, e.g., Hautus and Heymann [13, Theorems 5.10, 5.13]) that if L is any $m \times n$ constant matrix, then

$$(zI - (F - GL))^{-1}G = W(Q + LW)^{-1}.$$

Thus, for a feedback matrix L , the polynomial matrix $\hat{Q} := Q + LW$ corresponds to the pair $(F - GL, G)$. For the proof of the main theorem of this section which establishes an explicit connection between the different polynomial characterizations of (F_Q, G_Q) -invariant subspaces, we will need a modified version of a result by Fuhrmann [9, Theorem 4.3]. This result provides a natural K -isomorphism between K_Q and $K_{\hat{Q}}$ considered as vector spaces. Here we give a simpler new proof of a slightly corrected version of this result where we assume that Q^{-1} is proper. (Note that if Q is either row or column proper, this condition is automatically satisfied.)

Define the map

$$T_{Q\hat{Q}}: K_{\hat{Q}} \rightarrow K_Q: x \mapsto (Q\hat{Q}^{-1}x),$$

To see that the map $T_{Q\hat{Q}}$ is well defined, we need to show that $(Q\hat{Q}^{-1}x)$ belongs to K_Q for a given x in $K_{\hat{Q}}$. Let y be the unique polynomial vector and q be the unique strictly proper vector such that

$$Q\hat{Q}^{-1}x = y + q.$$

Since x belongs to $K_{\hat{Q}}$, $\hat{Q}^{-1}x$ is strictly proper and since Q^{-1} is assumed to be proper, $Q^{-1}q$ is also strictly proper. Hence, $(Q\hat{Q}^{-1}x)$ belongs to K_Q and the map $T_{Q\hat{Q}}$ is well defined.

Lemma 3.1: Let Q and \hat{Q} be $m \times m$ nonsingular polynomial matrices such that Q^{-1} is proper and

$$\hat{Q} = Q + LW$$

for some $m \times n$ constant matrix L . Then $T_{Q\hat{Q}}$ is a K -vector space isomorphism.

Proof: It is known (see Fuhrmann [7, Corollary (4.9)]) that the dimension of the K -vector space K_Q is the same as the degree of $\det Q$. But the degrees of $\det Q$ and $\det \hat{Q}$ are the same. It follows that dimensions of K_Q and $K_{\hat{Q}}$ are the same. Furthermore, it is clear that $T_{Q\hat{Q}}$ is a K -linear mapping. Thus, we only need to prove that $T_{Q\hat{Q}}$ is one to one. If x in $K_{\hat{Q}}$ is such that

$$T_{Q\hat{Q}}(x) = 0$$

then by definition $(Q\hat{Q}^{-1}x)$ is a strictly proper vector. Furthermore, it is clear that

$$Q\hat{Q}^{-1} = I + \hat{G}$$

where \hat{G} is strictly proper. It then follows that x is strictly proper. This yields $x = 0$. \square

Remark 3.2: Lemma 3.1 is stated in Fuhrmann [8] without the assumption that Q^{-1} is proper. We will now show by a counterexample that Lemma 3.1 is not necessarily true for arbitrary Q . Let us choose

$$Q = \begin{bmatrix} z^3 + 1 & z \\ z^3 + z^2 & z + 1 \end{bmatrix}.$$

It can be easily checked that Q^{-1} is not proper. Let us choose

$$\hat{Q} = \begin{bmatrix} z^3 + 1 & z \\ z^3 & z \end{bmatrix}$$

which is feedback equivalent to Q . Then we have

$$\hat{Q}^{-1} = \begin{bmatrix} 1 & -1 \\ -z^2 & z^2 + z^{-1} \end{bmatrix}$$

which is not proper. If we choose

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in $K_{\hat{Q}}$, then

$$T_{Q\hat{Q}}(x) = (Q\hat{Q}^{-1}x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

But we also have

$$Q^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (z+1)^{-1} \\ -z^{-1} \end{bmatrix}.$$

Thus, $T_{Q\hat{Q}}(x)$ does not belong to K_Q and, hence, the map $T_{Q\hat{Q}}$ is not well defined. This example shows that $T_{Q\hat{Q}}$ is not well defined in general, unless Q^{-1} is proper.

Lemma 3.1 is very useful since it provides a natural connection between the state-spaces K_Q and $K_{\hat{Q}}$ of WQ^{-1} and $W\hat{Q}^{-1}$, respectively. In the following theorem we establish an explicit link between the polynomial characterizations of (F_Q, G_Q) -invariant subspaces given in [6] and [10].

Theorem 3.3: Let V be an $m \times q$ polynomial matrix such that its columns are K -linearly independent. Let Q^{-1} be strictly proper. Then the following statements are equivalent.

- $Sp_K V$ is an (F_Q, G_Q) -invariant subspace.
- [6, Theorem (3.1)] There exist matrices H_1 in $K^{m \times q}$, H_2 in $K^{m \times q}$, and F_1 in $K^{q \times q}$ such that
 - $QH_1 + H_2 = V(zI - F_1)$.
- There exists an $m \times q$ polynomial matrix \hat{V} and there exist matrices H_1 in $K^{m \times q}$, F_1 in $K^{q \times q}$, and L in $K^{m \times n}$ such that if we define

$$\hat{Q} = Q + LW$$

then we have

- $T_{Q\hat{Q}}(\hat{V}) = V$ and
- $\hat{Q}H_1 + \hat{V}(zI - F_1)$

(i.e., the image of the subspace $Sp_K V$ under the map $T_{Q\hat{Q}}$ is $Sp_K \hat{V}$ which is $F_{\hat{Q}}$ -invariant).

- [10, Theorem 4.2] There exists an $m \times m$ nonsingular polynomial matrix Q_2 and an $m \times n$ constant matrix L such that for

$$\hat{Q} = Q + LW$$

- $\hat{Q}Q_2^{-1}$ is a polynomial matrix (i.e., Q_2 is a right nonsingular factor of \hat{Q}) and
- $Sp_K V = T_{Q\hat{Q}}(\hat{Q}Q_2^{-1}K_{\hat{Q}})$.

Proof: The equivalence between a) and b) is established in [6] and the equivalence between a) and d) is established in [10]. Here we establish the equivalence between b), c), and d).

b) \leftrightarrow c): We have

$$QH_1 + H_2 = V(zI - F_1).$$

Multiplying on the left by WQ^{-1} we get

$$WH_1 + WQ^{-1}H_2 = WQ^{-1}V(zI - F_1). \quad (3.4)$$

Taking polynomial parts on both sides of (3.4) we obtain

$$WH_1 = (WQ^{-1}V)(zI - F_1) + (WQ^{-1}V)_{-1}.$$

Since (F, G) is a reachable pair and W and Q are right coprime, it follows from [7, Theorems 4.5 and 4.7] that the map

$$\chi: K_Q \rightarrow K_{(zI - F)}: x \mapsto \pi_{(zI - F)}(Gx)$$

is a $K[z]$ -module isomorphism. Furthermore

$$\begin{aligned}\chi(x) &= \pi_{(zI - F)}(Gx) = (zI - F)((zI - F)^{-1}Gx) \\ &= (zI - F)(WQ^{-1}x)\end{aligned}$$

and since $\chi(x)$ is a constant vector it follows that

$$\chi(x) = (WQ^{-1}x)_{-1}.$$

Thus, $V_1 := (WQ^{-1}V)_{-1}$ is the image of V under the $K[z]$ -module isomorphism χ and, hence, $Sp_K V_1$ is an (F, G) -invariant subspace. Since the columns of V are K -linearly independent, the columns of the constant matrix V_1 are K -linearly independent. Further remembering that

$$F_Q(V) = zV - QH_1 = VF_1 + H_2,$$

b-i) is just a rewriting of the equation

$$FV_1 = V_1 F_1 + GH_2.$$

If we define L by

$$LV_1 = H_2$$

then we obtain

$$(F - GL)V_1 = V_1 F_1$$

i.e., the feedback matrix L makes $Sp_K V_1$ an $(F - GL)$ -invariant subspace. At least one such L exists since the columns of V_1 are K -linearly independent. With this choice of L we have

$$LWH_1 = L(WQ^{-1}V) \cdot (zI - F_1) + H_2. \quad (3.5)$$

Adding (3.5) and b-i) we get

$$(Q + LW)H_1 = (V + L(WQ^{-1}V)) \cdot (zI - F_1).$$

If we define

$$\hat{V} := V + L(WQ^{-1}V), \quad T_{Q\hat{Q}}^{-1}(V)$$

then we get

$$\hat{Q}H_1 = \hat{V}(zI - F_1).$$

Since $T_{Q\hat{Q}}^{-1}$ is the inverse of $T_{Q\hat{Q}}$, we also have

$$V = T_{Q\hat{Q}}^{-1}(\hat{V}).$$

c) \rightarrow b): We have

$$\hat{Q}H_1 = \hat{V}(zI - F_1).$$

Multiplying both sides by $\hat{Q}\hat{Q}^{-1}$ on the left and taking polynomial parts, we get

$$QH_1 = (Q\hat{Q}^{-1}\hat{V}) \cdot (zI - F_1) + (Q\hat{Q}^{-1}\hat{V})_{-1}.$$

Define

$$H_2 := -(Q\hat{Q}^{-1}\hat{V})_{-1} = L(W\hat{Q}^{-1}\hat{V})_{-1}.$$

Then we have

$$QH_1 + H_2 = V(zI - F_1).$$

This establishes the equivalence between b) and c).

c) \rightarrow d): Let L be as in c). Let Q_2, S_2 be a pair of relatively prime polynomial matrices such that

$$H_1(zI - F_1)^{-1} = Q_2^{-1}S_2.$$

Since the columns of V are K -linearly independent and $T_{Q\hat{Q}}^{-1}$ is an isomorphism, the columns of the matrix V are also K -linearly independent. Now, by c-ii) the pair (H_1, F_1) is observable. By Lemma 2.3, S_2 is a basis matrix for K_{Q_2} as a K -linear space. Then c-i) and Lemma 2.7 imply that Q_2 is a right factor of Q . Finally

$$V = T_{Q\hat{Q}}^{-1}(\hat{Q}Q_2^{-1}S_2)$$

which implies

$$Sp_K V = T_{Q\hat{Q}}^{-1}(\hat{Q}Q_2^{-1}K_{Q_2}).$$

d) \rightarrow c): Lemma 2.7 and d-i) imply that there exists a polynomial matrix \hat{V} and constant matrices H_1, F_1 such that

$$\hat{Q}H_1 = \hat{V}(zI - F_1)$$

where

$$H_1(zI - F_1)^{-1} = Q_2^{-1}S_2$$

for some polynomial matrix S_2 left coprime with Q_2 (which by Lemma 2.3 constitutes a basis matrix for K_{Q_2} as a K -linear space). It is given that

$$V = T_{Q\hat{Q}}^{-1}(\hat{Q}Q_2^{-1}S_2).$$

But we have

$$\hat{Q}Q_2^{-1}S_2 = \hat{V}.$$

Hence the proof. \square

Remark 3.6: The proof of Theorem 3.3 suggests a procedure for characterizing the set of all feedback matrices L which make $Sp_K \hat{V}$ and F_Q -invariant subspace (corresponding to the $(F - GL)$ -invariant subspace $Sp_K(WQ^{-1}V)_{-1}$) such that the matrix representation of F_Q^* restricted to $Sp_K \hat{V}$ is F_1 with the columns of \hat{V} as a basis for $Sp_K \hat{V}$. In particular

$$QH_1 + H_2 = V(zI - F_1)$$

and

$$(Q + LW)H_1 = \hat{V}(zI - F_1)$$

if and only if L is a solution to

$$L(WQ^{-1}V)_{-1} = H_2.$$

As shown in the proof of the preceding theorem this choice of L leads to the equation

$$F(WQ^{-1}V)_{-1} = (WQ^{-1}V)_{-1}F_1 + GL(WQ^{-1}V)_{-1}$$

which makes $(WQ^{-1}V)_{-1}$ an $(F - GL)$ -invariant subspace. Thus we first calculate the full column rank constant

matrix $(WQ^{-1}V)^{-1}$ and then solve for L in the linear equations over the field K :

$$L(WQ^{-1}V)^{-1} = H_2.$$

Remark 3.7: Now we will show that there is a correspondence between the pairs (H_1, F_1) and (F_Q, G_Q) -invariant subspaces and we will obtain a parametrization for (F_Q, G_Q) -invariant subspaces in terms of the pairs (H_1, F_1) . For this we first show that we can associate an (F_Q, G_Q) -invariant subspace with a given pair (H_1, F_1) . Given a pair (H_1, F_1) we will associate with it a polynomial matrix V and a constant matrix H_2 such that

$$QH_1 + H_2 = V(zI - F_1).$$

Then by Lemma 2.4 $Sp_K V$ will be an (F_Q, G_Q) -invariant subspace. To do this, first we express Q as

$$Q = Q_l z^l + Q_{l-1} z^{l-1} + \dots + Q_0$$

and then define

$$-H_2 := Q_l H_1 F_1^l + Q_{l-1} H_1 F_1^{l-1} + \dots + Q_0 H_1$$

(which is called the right functional value of QH_1 at F_1). Then by the generalized Bezout theorem (see Gantmacher [1958, ch. 4]), $(QH_1 + H_2)$ is divisible on the right by $(zI - F_1)$. Therefore, there exists a polynomial matrix V such that

$$QH_1 + H_2 = V(zI - F_1).$$

In fact, since $H_2(zI - F_1)^{-1}$ is strictly proper, the polynomial matrix V is given as

$$V = (QH_1(zI - F_1)^{-1})_+.$$

Conversely, given a polynomial matrix V such that $Sp_K V$ is an (F_Q, G_Q) -invariant subspace the matrices H_1 and F_1 are given by

$$F_Q(V) = \pi_Q(zV) = zV - QH_1$$

and

$$F_Q(V) = VF_1 + H_2$$

where the last equation states the fact that $Sp_K V$ is an (F_Q, G_Q) -invariant subspace.

However, this correspondence does not completely characterize (F_Q, G_Q) -invariant subspaces, since for a given pair (H_1, F_1) , the polynomial matrix

$$V = (QH_1(zI - F_1)^{-1})_+$$

may not have K -linearly independent columns. Also, two different pairs (H_1, F_1) and (\hat{H}_1, \hat{F}_1) may give rise to corresponding polynomial matrices V, \hat{V} such that $Sp_K V = Sp_K \hat{V}$. We shall first characterize those pairs of matrices H_1 in $K^{m \times q}$ and F_1 in $K^{q \times q}$ that correspond to an $m \times q$ polynomial matrix V having K -linearly independent col-

umns, for any arbitrary but fixed q . It is sufficient to consider such pairs of matrices (H_1, F_1) , since for a given q -dimensional (F_Q, G_Q) -invariant subspace Ψ , we can find an $m \times q$ polynomial matrix V and H_1, H_2 in $K^{m \times q}$ and F_1 in $K^{q \times q}$ such that

$$QH_1 + H_2 = V(zI - F_1)$$

and

$$Sp_K V = \Psi.$$

Such a V , clearly, has K -linearly independent columns. Let us express V as

$$V = V_{l-1} z^{l-1} + V_{l-2} z^{l-2} + \dots + V_0.$$

It now follows that

$$\begin{bmatrix} V_{l-1} \\ V_{l-2} \\ \vdots \\ V_0 \end{bmatrix} \begin{bmatrix} Q_l & 0 & \dots & 0 \\ Q_{l-1} & Q_l & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Q_1 & Q_2 & \dots & Q_l \end{bmatrix} \begin{bmatrix} H_1 \\ H_1 F_1 \\ \vdots \\ H_1 F_1^{l-1} \end{bmatrix} = 0. \quad (3.8)$$

Furthermore, V has full column rank (over K) if and only if the left-hand side of (3.8) has full column rank. If (h_{ij}) and (f_{ij}) are the entries of the matrices H_1 and F_1 , then the entries of V are polynomial functions of (h_{ij}) and (f_{ij}) given by (3.8). Let $\nu(H_1, F_1)$ denote an N -tuple with $q \times q$ minors of the left-hand side matrix of (3.8) as its entries.

(These are called the Plücker coordinates. Here $N = \binom{lm}{q}$.) Thus, the Plücker coordinates of $\nu(H_1, F_1)$ are polynomial functions of (h_{ij}) and (f_{ij}) . It now follows that V is full column rank iff $\nu(H_1, F_1)$ is not the zero vector. Thus, the pairs (H_1, F_1) that give full column rank V correspond to the complement of an algebraic set. Furthermore, two such pairs (H_1, F_1) and (\hat{H}_1, \hat{F}_1) give the same (F_Q, G_Q) -invariant subspace if and only if there exists a constant nonsingular $q \times q$ matrix α satisfying

$$V = \hat{V}\alpha.$$

This can happen if and only if

$$\nu(H_1, F_1) = \nu(\hat{H}_1, \hat{F}_1) \det \alpha.$$

Hence, the class of pairs H_1 is $K^{m \times q}$ and F_1 in $K^{q \times q}$ that give rise to distinct q -dimensional (F_Q, G_Q) -invariant subspaces correspond to the complement of an algebraic set modulo the equivalence " \sim ":

$$(H_1, F_1) \sim (\hat{H}_1, \hat{F}_1) \text{ if and only if } \nu(H_1, F_1) = \nu(\hat{H}_1, \hat{F}_1)c$$

where c is some nonzero constant.

This constitutes a complete parametrization of q -dimensional (F_Q, G_Q) -invariant subspaces in terms of pairs (H_1, F_1) .

Remark 3.9: Let $Z := PQ^{-1}$ be a $p \times m$ strictly proper transfer matrix, where P and Q are polynomial matrices and Q^{-1} is proper, and let (F_Q, G_Q, H_Q) be the Q -realiza-

tion of Z . We will now show explicitly the relation between (F_Q, G_Q) -invariant subspace in $\ker H_Q$ and common right nonsingular factors of P and \hat{Q} where \hat{Q} is feedback equivalent to Q . We will also show that cancelling the common right nonsingular factors of P and \hat{Q} corresponds to making an (F_Q, G_Q) -invariant subspace in $\ker H_Q$ unobservable by state feedback.

By Lemma 2.4 an $m \times q$ polynomial matrix V with K -linearly independent columns spans an (F_Q, G_Q) -invariant subspace in $\ker H_Q$ if and only if there exist constant matrices H_1, H_2 , and F_1 and a polynomial matrix V_1 such that

$$\begin{aligned} QH_1 + H_2 &= V(zI - F_1) \\ PH_1 &= V_1(zI - F_1). \end{aligned}$$

By Theorem 3.2, there exists an $m \times m$ nonsingular polynomial matrix \hat{Q} feedback equivalent to Q such that

$$\hat{Q}H_1 = \hat{V}(zI - F_1)$$

where

$$\hat{V} = T_{\hat{Q}Q^{-1}}(V).$$

Let Q_2 and S_2 be left coprime polynomial matrices such that

$$H_1(zI - F_1)^{-1} = Q_2^{-1}S_2.$$

Then by Lemma 2.7 we see that Q_2 is a common right divisor of P and \hat{Q} . Thus, $Sp_K V$ is an (F_Q, G_Q) -invariant subspace in $\ker H_Q$ only if Q_2 is a common right divisor of P and the subspace $T_{\hat{Q}Q^{-1}}(\hat{Q}Q_2^{-1}K_{Q_2})$ is made unobservable by state feedback.

For the converse, we assume that the rank [over $K(z)$] of P is m . Let Q_2 be a right factor of P . Then by Lemma 2.7, there exists a polynomial matrix S_2 and constant matrices H_1 and F_1 such that

$$Q_2^{-1}S_2 = H_1(zI - F_1)^{-1}$$

and

$$PH_1 = V_1(zI - F_1)$$

for some polynomial matrix V_1 . Then, as in Remark 3.7, we can find a polynomial matrix V for which there exists a constant matrix H_2 satisfying

$$QH_1 + H_2 = V(zI - F_1).$$

Then, by Lemma 2.4, $Sp_K V$ is an (F_Q, G_Q) -invariant subspace in $\ker H_Q$. Note that we can also find a feedback equivalent matrix \hat{Q} such that Q_2 is a common right divisor of \hat{Q} . Thus, we have explicitly shown the relation between (F_Q, G_Q) -invariant subspaces in $\ker H_Q$ and common right nonsingular divisors of P and \hat{Q} where \hat{Q} is feedback equivalent to Q . The results given above make contact with the geometric theory of linear systems and generalize the work of Wolovich [23].

IV. THE LARGEST (F, G) -INVARIANT AND REACHABILITY SUBSPACES IN $\ker H$ AND THE SYSTEM MATRIX

Let Z be a $p \times m$ strictly proper transfer matrix with the associated system matrix

$$T = \begin{bmatrix} Q & R \\ -P & U \end{bmatrix}$$

where P, Q, R , and U are $p \times r, r \times r, r \times m$, and $p \times m$ polynomial matrices such that $Z = PQ^{-1}R + U$ and $Q^{-1}R$ is strictly proper. Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization of Z associated with the system matrix T . In this section we will be concerned only with the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$, denoted by Ψ_M , and the largest reachability subspace contained in $\ker H_Q$, denoted by Ψ_R . In particular, if \hat{Q} is an $(r+p) \times (r+p)$ nonsingular matrix such that $\hat{Q}^{-1}T$ is a strictly proper transfer matrix, and $\Sigma_{\hat{Q}} = (F_{\hat{Q}}, G_{\hat{Q}}, H_{\hat{Q}})$ is the \hat{Q} -realization of $\hat{Q}^{-1}T$, then it will be shown that Ψ_M and Ψ_R are $K[z]$ -module isomorphic with the largest $(F_{\hat{Q}}, G_{\hat{Q}})$ -invariant subspace in $\ker H_{\hat{Q}}$ (which by Lemma 2.4 is K_I) and the largest reachability subspace in $\ker H_{\hat{Q}}$, respectively. This result will then be utilized in Section V to obtain a generalization of a theorem of Moore-Silverman on transmission polynomials. (For details see Section V.) By Lemma 2.1, $(F_{\hat{Q}}, H_{\hat{Q}})$ is an observable pair. Thus, for any system (F, G, H) , the largest (F, G) -invariant subspace in $\ker H$ and the largest reachability subspace in $\ker H$ are isomorphic to corresponding subspaces of an observable system. Using these results we also give a new constructive procedure to obtain the subspace Ψ_R for the case $Z = PQ^{-1}R + U$.

By Lemma 2.4 there exist constant matrices $H_1 = (\hat{H}_1' \hat{H}_2')'$ and F_1 such that the columns of the polynomial matrix $TH_1(zI - F_1)^{-1}$ constitute a basis for the K -linear space K_T . Furthermore, if π denotes the K -linear map

$$\pi: K_T \rightarrow K_Q: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x$$

then $\pi(K_T)$ is the largest (F_Q, G_Q) -invariant subspace contained in $\ker H_Q$. Let us define

$$\Phi = TH_1(zI - F_1)^{-1}.$$

We now define a $K[z]$ -module structure on K_T in the following way. Let x be an element of K_T . Since the columns of the matrix Φ constitute a basis for K_T , there exists a unique constant vector g such that

$$x = \Phi g.$$

We now define the scalar multiplication by z by the rule

$$\begin{aligned} z \cdot x &= \Phi F_1 g = TH_1(zI - F_1)^{-1} F_1 g \\ &= T(zH_1(zI - F_1)^{-1} g). \end{aligned}$$

It is clear that this definition of scalar multiplication by z gives K_T a $K[z]$ -module structure. Now we will establish

the isomorphism between K_I and the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$ in the following.

Theorem 4.1: The map π

$$\pi: K_I \rightarrow K_Q: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1$$

is a one-to-one K -linear map and image of π is Ψ_M .

Proof: It is obvious that π is K -linear. If π is not one-to-one there exists a nonzero vector x in K_I such that

$$\pi(x) = 0.$$

Since columns of Φ constitute a basis for K_I , there exists a unique constant vector g such that

$$x = \Phi g.$$

Let us partition H_1 and x in accordance with T as

$$\begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Q & R \\ -P & U \end{bmatrix} \begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix} (zI - F_1)^{-1} g$$

and

$$\pi(x) = x_1 - Q\hat{H}_1(zI - F_1)^{-1}g + R\hat{H}_2(zI - F_1)^{-1}g = 0.$$

Multiplying by PQ^{-1} on the left we get

$$P\hat{H}_1(zI - F_1)^{-1}g + PQ^{-1}R\hat{H}_2(zI - F_1)^{-1}g = 0$$

which can be rewritten as

$$P\hat{H}_1(zI - F_1)^{-1}g - U\hat{H}_2(zI - F_1)^{-1}g + Z\hat{H}_2(zI - F_1)^{-1}g = 0.$$

It now follows that

$$x_2 = Z\hat{H}_2(zI - F_1)^{-1}g.$$

Since x_2 is a polynomial vector and $Z\hat{H}_2(zI - F_1)^{-1}g$ is a strictly proper vector, we must have

$$x_2 = 0.$$

Thus we have

$$x = 0.$$

Using Lemma 2.4, $\text{im } \pi = \Psi_M$. \square

Remark 4.2: Let us define

$$V := \pi(\Phi)$$

(i.e., V is the upper part of the matrix Φ partitioned according to the partition of T). Then the columns of the polynomial matrix V constitute a basis for the K -linear space Ψ_M since π is one-to-one and $\text{im } \pi$ is Ψ_M . Let π_1 be defined as

$$\pi_1: K_I \rightarrow \Psi_M: x \mapsto \pi(x)$$

i.e., $\pi_1(x) = \pi(x)$. Now, if x_1 is an element of Ψ_M then there exists a unique constant vector g such that

$$x_1 = Vg$$

and if we define

$$x := \Phi g$$

then x is the unique constant vector in K_I such that

$$x_1 = \pi_1(x).$$

These considerations allow us to define $K[z]$ -module structure on Ψ_M in the following natural way. Define

$$z \cdot x_1 = \pi_1(z \cdot x) = \pi_1(\Phi F_1 g) = V F_1 g$$

which makes π_1 a $K[z]$ -module homomorphism. Thus, π_1 is a $K[z]$ -module isomorphism between K_I and Ψ_M .

Remark 4.3: We can obtain a basis matrix for the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$ as follows. Let M be an $(r+p) \times (r+p)$ unimodular matrix such that

$$MT = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}$$

where T_1 is an $l \times (r+m)$ row-proper polynomial matrix. Let $\alpha_1, \alpha_2, \dots, \alpha_l$ be the row degrees of T_1 . Define

$$\hat{V}_j := [z_j^{\alpha_1-1}, z_j^{\alpha_2-2}, \dots, 1]$$

and

$$\hat{V} := \text{diag}(\hat{V}_1, \hat{V}_2, \dots, \hat{V}_l).$$

Then it is shown in [6, Corollary (7.6)] that the columns of the polynomial matrix

$$\Phi := M^{-1} \begin{bmatrix} \hat{V} \\ 0 \end{bmatrix}$$

constitute a basis for the K -linear space K_I . If we define

$$V_1 := \pi_1(\Phi)$$

then by Lemma 4.1 the columns of the matrix V_1 constitute a basis for the K -linear space Ψ_M .

In the next theorem we show how the largest reachability subspace Ψ_R contained in $\ker H_Q$ can be obtained from K_I .

Theorem 4.4: Let G_1 be a constant matrix such that columns of ΦG_1 span the K -linear space $K_I \cap \text{Sp}_K T$. Then we have

$$\Psi_R = \pi_1(\text{Sp}_K \Phi [G_1, F_1 G_1, \dots, F_1^{q-1} G_1]).$$

Proof: Let us first define

$$V := \pi_1(\Phi).$$

If G_2 is a constant matrix such that the columns of $V G_2$ span the K -linear space $\Psi_M \cap \text{Sp}_K R$, then we have by Lemma 2.5

$$\Psi_R = \text{Sp}_K (V [G_2, F_1 G_2, \dots, F_1^{q-1} G_2]).$$

Thus, if we prove that the columns of VG_1 span the K -linear space $\Psi_M \cap Sp_A R$, then

$$\begin{aligned}\Psi_R &= Sp_A(V[G_1, F_1G_1, \dots, F_1^{q-1}G_1]) \\ &= Sp_A(\pi_1\Phi[G_1, F_1G_1, \dots, F_1^{q-1}G_1]) \\ &= \pi_1(Sp_A\Phi[G_1, F_1G_1, \dots, F_1^{q-1}G_1]).\end{aligned}$$

So let x_1 be an arbitrary element of $\Psi_M \cap Sp_A R$. Then there exists a constant vector d_1 such that

$$x_1 = Rd_1$$

and a unique constant vector g such that

$$x_1 = Vg.$$

Let us define

$$x = \Phi g = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for some polynomial vector x_2 . Since x belongs to K_T there exist strictly proper vectors q_1 and q_2 such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

It follows that

$$x_1 = Qq_1 + Rq_2 = Rd_1.$$

Multiplying by PQ^{-1} we have

$$Pq_1 + PQ^{-1}Rq_2 = PQ^{-1}Rd_1$$

which can be rewritten as

$$-Pq_1 + Uq_2 = Z(q_2 - d_1) + Ud_1.$$

We have

$$x_2 = Uq_2 - Pq_1$$

and consequently

$$x_2 = Z(q_2 - d_1) + Ud_1.$$

Since Z is strictly proper and $(q_2 - d_1)$ is proper, it follows that

$$x_2 = Ud_1$$

and we have

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} R \\ U \end{bmatrix} d_1 = T \begin{bmatrix} 0 \\ d_1 \end{bmatrix}.$$

Consequently, x belongs to the K -linear space $Sp_A T$ and also to K_T , and hence belongs to $Sp_A T \cap K_T$. Furthermore, since the columns of ΦG_1 span the K -linear space $K_T \cap Sp_A T$, there exists a constant vector c such that

$$x = \Phi G_1 c$$

and hence

$$x_1 = VG_1 c$$

(i.e., x_1 belongs to the K -linear space $Sp_A VG_1$).

Conversely, let x_1 be an arbitrary element of $Sp_A VG_1$. We will prove that x_1 also belongs to $\Psi_M \cap Sp_A R$. Let c be a constant vector such that

$$x_1 = VG_1 c.$$

If we let

$$x = \Phi G_1 c$$

then x belongs to $K_T \cap Sp_A T$ and, therefore, there exist constant vectors d_1 and d_2 such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} Q & R \\ P & U \end{bmatrix} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}.$$

It follows that

$$x_1 = Qd_2 + Rd_1.$$

Therefore

$$d_2 = Q^{-1}x_1 - Q^{-1}Rd_1.$$

Since x_1 belongs to K_Q and $Q^{-1}R$ is strictly proper, $Q^{-1}x_1 - Q^{-1}Rd_1$ is also strictly proper. Consequently

$$d_2 = 0.$$

Thus

$$x_1 = Rd_1 = \pi_1(x)$$

and therefore x_1 belongs to $\Psi_M \cap Sp_A R$. This proves that columns of VG_1 span the K -linear space $\Psi_M \cap Sp_A R$. This completes the proof of the theorem. \square

Remark 4.5: Let \hat{Q} be a $(p+r) \times (p+r)$ nonsingular polynomial matrix such that $\hat{Q}^{-1}T$ is a strictly proper rational matrix. (It is obvious that such matrices \hat{Q} exist.) Let $\Sigma_{\hat{Q}} := (F_{\hat{Q}}, G_{\hat{Q}}, H_{\hat{Q}})$ be the \hat{Q} -realization associated with the strictly proper transfer matrix $\hat{Q}^{-1}T$. By Lemma 2.4, K_T is the largest $(F_{\hat{Q}}, G_{\hat{Q}})$ -invariant subspace in $\ker H_{\hat{Q}}$. Furthermore, if G_1 is as in Theorem 4.4, then $Sp_A \Phi[G_1, F_1G_1, \dots, F_1^{q-1}G_1]$ is the largest reachability subspace contained in $\ker H_{\hat{Q}}$ denoted by Ψ_T . With these interpretations for K_T and Ψ_T and using Theorems 4.1 and 4.2, we have the following fact:

$$\pi_1: K_T \rightarrow \Psi_M$$

is a $K[z]$ -module isomorphism such that

$$1) \quad \pi_1(K_T) = \Psi_M$$

and

$$2) \quad \pi_1(\Psi_T) = \Psi_R.$$

Thus, π_1 is a K -linear map that maps the largest $(F_{\hat{Q}}, G_{\hat{Q}})$ -invariant subspace in $\ker H_{\hat{Q}}$, and the largest reachability subspace contained therein onto the largest $(F_{\hat{Q}}, G_{\hat{Q}})$ -invariant subspace in $\ker H_{\hat{Q}}$ and the largest reachability subspace contained therein, respectively. Recall that $(F_{\hat{Q}}, H_{\hat{Q}})$ is an observable pair. Thus, the largest (F, G) -invariant subspace in $\ker H$ and the largest reachability sub-

space contained in it can be viewed as the corresponding subspaces of an observable system, namely $\Sigma_{\hat{Q}}$.

Remark 4.6: It follows from the preceding remark that any constructive procedure for obtaining the largest reachability subspace contained in $\ker H_{\hat{Q}}$ for the \hat{Q} -realization of the strictly proper transfer matrix $\hat{Q}^{-1}T$ can be used to obtain the largest reachability subspace contained in $\ker H_Q$ for the Q -realization of the transfer matrix Z . A constructive procedure to obtain the largest reachability subspace contained in $\ker H_{\hat{Q}}$ for \hat{Q} -realization of $\hat{Q}^{-1}T$ is given [6, Theorem 7.7]. Using this procedure, we can obtain a polynomial matrix Φ_R whose columns constitute a basis for the largest reachability subspace contained in $\ker H_{\hat{Q}}$. Then by Theorem 4.4 and the preceding remark, it follows that the columns of the polynomial matrix $\pi_1(\Phi_R)$ constitute a basis for the K -vector space Ψ_R . Thus, we have a constructive procedure to obtain the largest reachability subspace Ψ_R for the Q -realization of Z .

V. A MODULE THEORETIC APPROACH TO TRANSMISSION POLYNOMIALS

Let T be a $\mu \times \nu$ polynomial matrix. Let H_1, F_1 be constant matrices such that the columns of $\Phi_1 = TH_1(zI - F_1)^{-1}$ constitute a basis for the K -vector space K_T . As in Section IV, we define a $K[z]$ -module structure on K_T . For a given x in K_T , let g be the unique constant vector such that $x = TH_1(zI - F_1)^{-1}g$. Then we define the scalar multiplication by z as

$$z \cdot x := T(zH_1(zI - F_1)^{-1}g) = TH_1(zI - F_1)^{-1}F_1g.$$

In case T is square and nonsingular, it is shown by Fuhrmann [7] that K_T is $K[z]$ -module isomorphic to $K^n[z]/TK^n[z]$ and, hence, the nonconstant invariant factors of T are the same as the invariant factors of the $K[z]$ -module K_T . In general, T may not be square and nonsingular. (For example, the system matrix is not necessarily square and nonsingular.) In this section we generalize this result using the generalization of K_T given in [6] (also see Section II), and prove that the nonconstant invariant factors of the quotient module $K_T / \langle K_T \cap \text{Sp}_K T \rangle$ are the same as the nonconstant invariant factors of the polynomial matrix T . This fundamental result also leads to the following generalization of a theorem of Moore and Silverman [19] (see Lemma 2.8 for the statement of this theorem) on transmission polynomials.

Let T be the $(r+p) \times (r+m)$ system matrix associated with the matrix fraction description $Z = PQ^{-1}R + U$, and let $L_Q: K_Q \rightarrow K^m$ be a K -linear map such that $(F_Q + G_Q L_Q)\Psi_M \subseteq \Psi_M$, where Ψ_M denotes the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$. Then the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on Ψ_M / Ψ_R are the same as the nonconstant invariant factors of the polynomial system matrix T . (See Theorem 5.5.)

For the case where $Q = (zI - F)$, $R = G$, and $P = H$, this theorem is also proved by Molinari [18] and by Anderson [1] for a canonical (F, G, H) . Our results, based on the natural $K[z]$ -module structure on Ψ_M and Ψ_R , unify and generalize the results in [7] and [19].

We also specialize our results to the left and right polynomial fractional representations which leads to simple characterizations of transmission polynomials in terms of numerator polynomial matrices. Finally, we derive a simple result which will be used in the next section to obtain polynomial characterizations of stabilizability subspaces.

To prove our main results of this section, first we establish the following lemmas.

Let the row rank [over the field $K(z)$] of the polynomial matrix T be $\gamma \leq \mu$. Then there exists a unimodular matrix M such that

$$MT = \begin{bmatrix} \hat{T} \\ 0 \end{bmatrix}$$

where \hat{T} is a full row rank [over the field $K(z)$] $\gamma \times \nu$ polynomial matrix. Consider the map $f: K_T \rightarrow K^\mu[z]/MTK^\nu[z]: x \mapsto Mx + MTK^\nu[z]$. Let L_1 and G_1 be constant matrices such that

$$\text{Sp}_K TL_1 = \text{Sp}_K \Phi G_1 = K_T \cap \text{Sp}_K T.$$

We will first prove that f is a $K[z]$ -module homomorphism with $\langle TL_1 \rangle$ as its kernel.

Lemma 5.1: f is a $K[z]$ -module homomorphism.

Proof: For x_1, x_2 in K_T , we have

$$f(x_1 + x_2) = Mx_1 + Mx_2 + MTK^\nu[z] = f(x_1) + f(x_2).$$

Let g be the unique constant vector such that

$$x_1 = \Phi g = TH_1(zI - F_1)^{-1}g.$$

Then we have

$$\begin{aligned} f(z \cdot x_1) &= f(TH_1(zI - F_1)^{-1}F_1g) \\ &= f(TH_1(zI - F_1)^{-1}g - TH_1g). \end{aligned}$$

It now follows that

$$f(z \cdot x_1) = zMx - MTH_1g + MTK^\nu[z] = z \cdot f(x_1).$$

Thus, f is a $K[z]$ -module homomorphism. \square

The kernel of the homomorphism f is obtained in the following.

Lemma 5.2: The kernel of f is the $K[z]$ -submodule $\langle TL_1 \rangle$.

Proof: It is clear that

$$f(TL_1) = MTL_1 + MTK^\nu[z] = 0.$$

Furthermore, since f is a $K[z]$ -module homomorphism, it follows that $f(\langle TL_1 \rangle) = 0$ and, hence, $\langle TL_1 \rangle \subseteq \ker f$.

Conversely, let x in K_T be such that $f(x) = 0$. Then there exists a polynomial vector b and a strictly proper vector q , such that $x = Tb = Tq$. Let b be represented as $b_0 + b_1z + \dots + b_rz^r$, where b_i are in K^r and b_r is nonzero. Furthermore, since x belongs to K_T , there exists a unique constant vector g such that $x = \Phi g$. Now we have

$$T \begin{pmatrix} 1 \\ \vdots \\ b_r z^r \\ 0 \end{pmatrix} = TH_1(zI - F_1)^{-1}g.$$

Multiplying by z^{-l} we get

$$Th_l = TH_l(zI - F)^{-1} z^{-l} g - \sum_{i=0}^{l-1} Th_i z^{i-l}.$$

Consequently, Th_l belongs to K_T as well as to $Sp_K T$ and hence to $Sp_K TL_1$. Furthermore, there exists a unique constant vector c such that

$$Th_l = \Phi c = TH_l(zI - F_1)^{-1} c.$$

Comparing coefficients in the two formal power series expansions for Th_l we get

$$Th_l = -TH_l F_1^{l-1} c, \quad 0 \leq j \leq l-1$$

$$TH_l(zI - F_1)^{-1} g = TH_l(zI - F_1)^{-1} F_1^l c.$$

It now follows that

$$\begin{aligned} x = Th &= Th_l z^l + \sum_{i=0}^{l-1} Th_i z^i \\ &= TH_l(zI - F_1)^{-1} F_1^l c = z^l \cdot Th_l. \end{aligned}$$

Consequently, since Th_l belongs to $Sp_K TL_1$, x belongs to $\langle TL_1 \rangle$. Thus, $f(x) = 0$ implies x belongs to $\langle TL_1 \rangle$. This proves that

$$\ker f = \langle TL_1 \rangle = \langle K_T \cap Sp_K T \rangle. \quad \square$$

Finally, the image of f is obtained in the following.

Lemma 5.3: Image of the map f is given by

$$\text{im } f = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} + MTK^r[z]; a \text{ in } K^r[z] \right\}.$$

Proof: For any x in K_T , there exists a strictly proper vector q such that $x = Tq$. Consequently, we have

$$\begin{aligned} f(x) &= Mx + MTK^r[z] \\ &= MTq + MTK^r[z] = \begin{bmatrix} \hat{T}q \\ 0 \end{bmatrix} + MTK^r[z]. \end{aligned}$$

Since Tq is a polynomial vector in $K[z]$, it follows that $f(x)$ belongs to

$$\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} + MTK^r[z]; a \text{ in } K^r[z] \right\}.$$

Conversely, since \hat{T} is full row rank, there exists a rational matrix S such that $\hat{T}S = I$. Now let a be in $K^r[z]$. Then we have

$$MTSa = \begin{bmatrix} \hat{T}Sa \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

Let l be the unique polynomial vector and q be the unique strictly proper vector such that $Sa = l + q$. It follows that

$$Tq = TSa - Tl = M^{-1} \begin{bmatrix} a \\ 0 \end{bmatrix} - Tl.$$

Therefore, Tq is polynomial and, hence, belongs to K_T . Furthermore

$$f(Tq) = MTq + MTK^r[z] = \begin{bmatrix} a \\ 0 \end{bmatrix} + MTK^r[z].$$

We have proved that

$$\text{im } f = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} + MTK^r[z]; a \text{ in } K^r[z] \right\}. \quad \square$$

The following theorem is an immediate consequence of the preceding lemmas, and it relates the $K[z]$ -module structures on K_T and the natural module structure on $K^r[z]/\hat{T}K^r[z]$.

Theorem 5.4: The quotient module $K_T/\langle K_T \cap Sp_K T \rangle$ is $K[z]$ -module isomorphic to $K^r[z]/\hat{T}K^r[z]$. Furthermore, the nonconstant invariant factors of the $K[z]$ -module $K_T/\langle K_T \cap Sp_K T \rangle$ are the same as the nonconstant invariant factors of the polynomial matrix T .

Proof: Let π_2 be the projection map

$$\begin{aligned} \pi_2: K^r[z]/MTK^r[z] &\rightarrow K^r[z]/\hat{T}K^r[z]; \\ \begin{bmatrix} a \\ b \end{bmatrix} + MTK^r[z] &\mapsto a + \hat{T}K^r[z]. \end{aligned}$$

It is clear the π_2 is a well-defined $K[z]$ -module homomorphism. Using the results of Lemmas 5.1, 5.2, and 5.3, it follows that the map π_2 is an onto $K[z]$ -module homomorphism with kernel $\langle K_T \cap Sp_K T \rangle$. Hence, by the fundamental homomorphism theorem (see Lang [17, ch. 3]), it follows that $K_T/\langle K_T \cap Sp_K T \rangle$ and $K^r[z]/\hat{T}K^r[z]$ are $K[z]$ -module isomorphic. Consequently, the nonconstant invariant factors of $K_T/\langle K_T \cap Sp_K T \rangle$ are the same as the nonconstant invariant factors of \hat{T} which are the same as the nonconstant invariant factors of T since

$$MT = \begin{bmatrix} \hat{T} \\ 0 \end{bmatrix}$$

and M is unimodular. This completes the proof of the theorem. \square

Theorem 5.4 is a generalization of a result by Fuhrmann [7] which states that if T is square and nonsingular, then K_T and $K^r[z]/TK^r[z]$ are $K[z]$ -module isomorphic. We will now apply the results of Theorem 5.4 to obtain a generalization of the theorem of Moore and Silverman [19]. In what follows, (F_Q, G_Q, H_Q) represents the Q -realization of $Z = PQ^{-1}R + U$.

Theorem 5.5: Let Ψ_M and Ψ_R , respectively, denote the largest (F_Q, G_Q) -invariant and the largest reachability subspace in $\ker H_Q$. Let $L_Q: K_Q \rightarrow K^m$ be a K -linear map such that

$$(F_Q + G_Q L_Q)\Psi_M \subseteq \Psi_M.$$

Then the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on Ψ_M/Ψ_R are the same as the nonconstant invariant factors of the polynomial system matrix T .

Proof: As noted in Lemma 2.4, there exist constant matrices H_1 and F_1 such that the columns of the polynomial matrix

$$\Phi = TH_l(zI - F_1)^{-1}$$

constitute a basis for the K -linear space K_T . Furthermore, by Theorems 4.1 and 4.4, it follows that the map

$$\pi_1: K_T \rightarrow \Psi_M: \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a$$

is a $K[z]$ -module isomorphism such that

$$\pi_1(\langle K_T \cap Sp_K T \rangle) = \Psi_R.$$

By Theorem 5.4, the nonconstant invariant factors of $K_T / \langle K_T \cap Sp_K T \rangle$ are the same as the nonconstant invariant factors of the polynomial system matrix T . Also, the module structure on Ψ_M and Ψ_R corresponds to the matrix F_1 . Hence, by Lemma 2.4 the module structure on Ψ_M / Ψ_R corresponds to the action of the linear map induced by $(F_Q + G_Q L_Q)$, and since $K_T / \langle K_T \cap Sp_K T \rangle$ and Ψ_M / Ψ_R are $K[z]$ -module isomorphic, it follows that the nonconstant invariant factors of T are the same as the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on Ψ_M / Ψ_R . \square

The next result is a simple corollary of the preceding theorem.

Corollary 5.6: Let Z be a $p \times m$ strictly proper transfer matrix with right (respectively, left) matrix fraction representation $Z = PQ^{-1}$ (respectively, $Z = Q^{-1}R$). Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization of Z . Let Ψ_M and Ψ_R , respectively, represent the largest (F_Q, G_Q) -invariant subspace and the largest reachability subspace contained in $\ker H_Q$. Let $L_Q: K_Q \rightarrow K^m$ be a K -linear map such that Ψ_M is $(F_Q + G_Q L_Q)$ -invariant. Then the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on Ψ_M / Ψ_R are the same as the nonconstant invariant factors of P (respectively, R).

Proof: By Theorem 5.5, the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on Ψ_M / Ψ_R are the same as the nonconstant invariant factors of the associated polynomial system matrix. If $Z = PQ^{-1}$, then the system matrix is given by

$$T = \begin{bmatrix} Q & I \\ -P & 0 \end{bmatrix}.$$

If we define

$$M = \begin{bmatrix} I & 0 \\ -Q & I \end{bmatrix}$$

then M is unimodular. Now we have

$$TM = \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix}$$

and, hence, the nonconstant invariant factors of T are the same as those of P . This completes the proof for the case $Z = PQ^{-1}$. Proof for the left matrix fraction description $Z = Q^{-1}R$ is similar. \square

We will now establish a result that will be useful in obtaining a characterization of the largest stabilizability subspace contained in $\ker H_Q$. (See Section VI, Theorem 6.1.)

Corollary 5.7: Let R be a $\mu \times \nu$ polynomial matrix and let the dimension of the K -linear space K_R be η . If H_1, F_1 are constant matrices such that the columns of $RH_1(zI - F_1)^{-1}$ constitute a basis for the K -linear space K_R , then the product

of invariant factors of R divides the characteristic polynomial of F_1 . Furthermore, if α is a given monic polynomial of degree η , which is divisible by the product of the invariant factors of R , then there exists a pair (H_1, F_1) such that the columns of $RH_1(zI - F_1)^{-1}$ constitute a basis for K_R and the characteristic polynomial of F_1 is α .

Proof: Let χ denote the product of invariant factors of R . Let us define $\Phi := RH_1(zI - F_1)^{-1}$. As in Remark 4.5, let Q be a $\mu \times \mu$ nonsingular polynomial matrix such that $Q^{-1}R$ is strictly proper. Let (F_Q, G_Q, H_Q) be the Q -realization of $Q^{-1}R$. Then by Lemma 2.4 it follows that K_R is the largest (F_Q, G_Q) -invariant subspace in $\ker H_Q$. Furthermore, there exists a K -linear map $L_Q: K_Q \rightarrow K^\nu$ such that K_R is $(F_Q + G_Q L_Q)$ -invariant, and F_1 is the matrix representation of $(F_Q + G_Q L_Q)$ restricted to K_R with the columns of Φ as a basis for K_R . Let Ψ_R represent the largest reachability subspace contained in $\ker H_Q$. By Corollary 5.6, it follows that the nonconstant invariant factors of R are the same as the nonconstant invariant factors of the linear map induced by $(F_Q + G_Q L_Q)$ on K_R / Ψ_R . Hence, χ divides the characteristic polynomial of $(F_Q + G_Q L_Q)$ restricted to K_R which is the same as the characteristic polynomial of F_1 .

Let β be defined by $\beta := \alpha / \chi$. It now follows from Wonham [25, Corollary 5.2] that there exists a K -linear map $L_Q: K_Q \rightarrow K^\nu$ such that K_R is $(F_Q + G_Q L_Q)$ -invariant, the characteristic polynomial of $(F_Q + G_Q L_Q)$ restricted to Ψ_R is β , and the characteristic polynomial of $(F_Q + G_Q L_Q)$ restricted to K_R is α . By Lemma 2.4, there exist constant matrices H_1 and F_1 such that the columns of $RH_1(zI - F_1)^{-1}$ constitute a basis for K_R and the characteristic polynomial of F_1 is α . This completes the proof. \square

VI. STABILIZABILITY SUBSPACES

Throughout this section we assume that the field K is the field of real numbers denoted by \mathbf{R} . In what follows a general type of stability is considered as in Wonham [25] and Hautus [11], [12]: we are given a subset C of the field of complex numbers \mathbf{C} , satisfying the condition that $C \cap \mathbf{R}$ is nonempty and C is symmetric about the real axis. A polynomial with real coefficients is said to be *stable* iff all of its roots are in C .

Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ be the Q -realization (see Section II) associated with the strictly proper $p \times m$ transfer matrix

$$Z = PQ^{-1}R + U.$$

A subspace M of \mathbf{R}_Q is said to be a *stabilizability subspace* (see Hautus [11], [12]) if and only if there exists a linear map

$$L_Q: \mathbf{R}_Q \rightarrow \mathbf{R}^m$$

such that M is $(F_Q + G_Q L_Q)$ -invariant and the characteristic polynomial of $(F_Q + G_Q L_Q)$ restricted to M is stable. It has been shown in Wonham [25] and Hautus [11], [12] that stabilizability subspaces are very useful in studying stability properties associated with system synthesis problems. In this section we establish a characterization of and give a

constructive procedure to obtain the largest stabilizability subspace of Σ_Q contained in $\ker H_Q$.

The following theorem is the main result of this section.

Theorem 6.1: Let T be the polynomial system matrix of the strictly proper transfer matrix

$$Z = PQ^{-1}R + U$$

where $Q^{-1}R$ is strictly proper. Let $\epsilon_1, \epsilon_2, \dots, \epsilon_q$ be the invariant factors of the polynomial matrix T . Let ϵ_i^+ and ϵ_i^- be such that

$$\epsilon_i = \epsilon_i^+ \epsilon_i^-, \quad i = 1, 2, \dots, q$$

where ϵ_i^+ is the unstable factor of ϵ_i and ϵ_i^- is the stable factor of ϵ_i . Let T^+ and T^- be $(p+r) \times (p+r)$ and $(p+r) \times (r+m)$ polynomial matrices such that

$$T = T^+ T^-$$

T^+ is nonsingular with invariant factors $\epsilon_1^+, \epsilon_2^+, \dots, \epsilon_q^+, 1, \dots, 1$, and the invariant factors of T^- are $\epsilon_1^-, \epsilon_2^-, \dots, \epsilon_q^-, 0, \dots, 0$. Then $\pi(T^+ R_T)$ is the largest stabilizability subspace contained in $\ker H_Q$.

Proof: It is clear that a stabilizability subspace is necessarily an (F_Q, G_Q) -invariant subspace. We shall first prove that $\pi(T^+ R_T)$ is a stabilizability subspace contained in $\ker H_Q$.

It follows by the definition of T^- that the invariant factors of T^- are stable. Now by Corollary 5.7 there exists an observable pair of matrices (H_1, F_1) such that the columns of the polynomial matrix

$$\Phi = T^- H_1 (zI - F_1)^{-1}$$

constitute a basis for the R -linear space R_T and the characteristic polynomial of F_1 is stable. Let us define

$$\Psi := T^+ \Phi = TH_1(zI - F_1)^{-1}.$$

Then we have

$$\pi(Sp_R \Psi) = \pi(T^+ R_T).$$

By Lemma 2.4 it follows that $\pi(T^+ R_T)$ is an (F_Q, G_Q) -invariant subspace in $\ker H_Q$ and there exists an R -linear map

$$L_Q: R_Q \rightarrow R^m$$

such that $\pi(T^+ R_T)$ is $(F_Q + G_Q L_Q)$ -invariant and the matrix representation of $(F_Q + G_Q L_Q)$ restricted to $\pi(T^+ R_T)$ is F_1 . Finally, since the characteristic polynomial of F_1 is stable, it follows that $\pi(T^+ R_T)$ is a stabilizability subspace contained in $\ker H_Q$.

We will now prove that any stabilizability subspace contained in $\ker H_Q$ is contained in $\pi(T^+ R_T)$. Let V be a polynomial matrix whose columns constitute a basis for a stabilizability subspace contained in $\ker H_Q$. Now, by the definition of stabilizability subspaces it follows that there exists an R -linear map

$$L_Q: R_Q \rightarrow R^m$$

such that $Sp_R V$ is $(F_Q + G_Q L_Q)$ -invariant and the characteristic polynomial of $(F_Q + G_Q L_Q)$ restricted to $Sp_R V$ is stable. If we choose the columns of V as a basis for $Sp_R V$, then there exist constant matrices \hat{H} and \hat{F} such that

$$\pi(\hat{H}(zI - \hat{F})^{-1}) = V$$

and the characteristic polynomial of \hat{F} is stable. Let us define Ψ as

$$\Psi := \hat{H}(zI - \hat{F})^{-1}.$$

It now follows that

$$(T^+)^{-1} \Psi = T^- \hat{H}(zI - \hat{F})^{-1}.$$

Now the denominator polynomials of the rational matrix $(T^+)^{-1} \Psi$ are unstable, whereas the denominator polynomials of the rational matrix $T^- \hat{H}(zI - \hat{F})^{-1}$ are stable. Hence, $T^- \hat{H}(zI - \hat{F})^{-1}$ must be a polynomial matrix. Thus we have

$$\Psi = T \hat{H}(zI - \hat{F})^{-1} = T^+ T^- \hat{H}(zI - \hat{F})^{-1} \subseteq T^+ R_T$$

and consequently

$$Sp_R \pi(\Psi) \subseteq \pi(T^+ R_T).$$

This completes the proof of the theorem. \square

Theorem 6.1 also provides a constructive procedure to obtain the largest stabilizability subspace contained in $\ker H_Q$. The procedure can be outlined as follows.

1) Using the invariant factor algorithm for polynomial matrices (see, for example, Lang [17, ch. 15]), find unimodular matrices M and N such that MTN is in Smith form with invariant factors $\epsilon_1, \dots, \epsilon_q$.

2) Define T^+ and T^- as follows:

$$T^+ := M^{-1} \text{diag}(\epsilon_1^+, \epsilon_2^+, \dots, \epsilon_q^+, 1, \dots, 1)$$

and

$$T^- := \begin{bmatrix} \epsilon_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \epsilon_2 & & & \vdots \\ & & \ddots & & \\ 0 & \cdots & & \epsilon_q & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \cdots & 0 \end{bmatrix} N^{-1}.$$

3) As described in Remark 4.3, construct a polynomial matrix Φ such that the columns of Φ constitute a basis for the R -linear space R_T .

4) Then the columns of the polynomial matrix $\pi(T^+ \Phi)$ constitute a basis for the largest stabilizability subspace contained in $\ker H_Q$.

Thus, Theorem 6.1 provides a characterization of and a constructive procedure to obtain the largest stabilizability subspace contained in $\ker H_Q$ in terms of the system matrix T .

ACKNOWLEDGMENT

The authors wish to thank Prof. R. E. Kalman for several discussions on the problems studied in this paper. The authors are also grateful to the reviewers for their constructive suggestions.

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